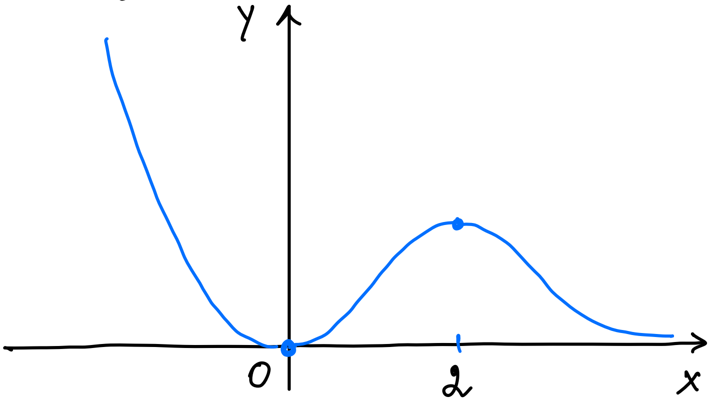
Lösningsskiss till tentamen i Matematisk analys, 764G07 del I.

2022-01-07.

1. $y(x) = x^4 e^{-2x}$. • $D_f = \mathbb{R}$; $\lim\limits_{x\to-\infty} x^4 e^{-2x} = \infty$, $\lim\limits_{x\to\infty} x^4 e^{-2x} = 0$



• $y' = 4x^3 e^{-2x} - 2x^4 e^{-2x} =$

$= 2x^3 e^{-2x}(2-x) = 0 \iff$

$x = 0$ eller $x = 2$

	$x<0$	0	$0<x<2$	2	$x>2$
x^3	−	0	+		+
$2-x$	+		+	0	−
y'	−	0	+	0	−
y	$\searrow$	l. min	$\nearrow$	l. max	$\searrow$

$y(0) = 0$ - lok. min

$y(2) = 16e^{-4}$ - lok. max

Svar: $y = 0$ - vågrät asymptot

lok. max. $y_{max} = y(2) = 16e^{-4}$

lok. min. $y_{min} = y(0) = 0$.

2a $p(z) = z^4 - 4z^3 + 4z^2 + 4z - 5 = 0 \Rightarrow p(z) = (z-1)(z+1)(z^2-4z+5)$

$z^2 - 4z + 5 = (z-2)^2 + 1 = 0 \iff z - 2 = \pm i \iff z = 2 \pm i$

Svar: $z_1 = 1$, $z_2 = -1$, $z_{3,4} = 2 \pm i$.

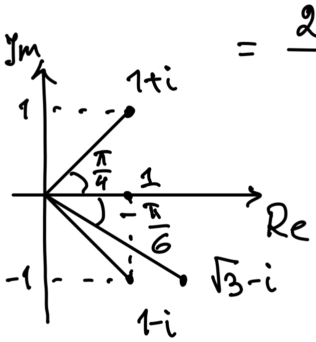
2b $z = \dfrac{(\sqrt{3}-i)^3(1-i)}{1+i}$; $|z| = \dfrac{|\sqrt{3}-i|^3 |1-i|}{|1+i|} =$

$|\sqrt{3}-i| = \sqrt{3+1} = 2$

$|1-i| = \sqrt{1+1} = \sqrt{2}$

$|1+i| = \sqrt{1+1} = \sqrt{2}$

$= \dfrac{2^3 \cdot \sqrt{2}}{\sqrt{2}} = 2^3 = 8.$



$\arg z = 3\arg(\sqrt{3}-i) + \arg(1-i) - \arg(1+i) =$

$= 3\cdot\left(-\frac{\pi}{6}\right) - \frac{\pi}{4} - \frac{\pi}{4} + 2\pi n = -\frac{\pi}{2} - \frac{\pi}{2} + 2\pi n = \pi + 2\pi n$

Svar: $|z| = 8$, $\arg z = \pi + 2\pi n$.

3a $\lim\limits_{x\to\infty}\left(\sqrt{x^2+9x} - x\right) = \lim\limits_{x\to\infty} \dfrac{x^2+9x-x^2}{\sqrt{x^2+9x}+x} = \lim\limits_{x\to\infty} \dfrac{9x}{x\left(\sqrt{1+\frac{9}{x}}+1\right)} = \dfrac{9}{2}$

3b

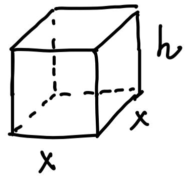
$$\lim_{x\to\infty}\left(\ln(5x+3)-\ln(2x-4)\right)=\lim_{x\to\infty}\ln\frac{5x+3}{2x-4}=\lim_{x\to\infty}\ln\frac{x\left(5+\frac{3}{x}\right)}{x\left(2-\frac{4}{x}\right)}$$
$$=\ln\frac{5}{2}.$$

3c

$$\lim_{x\to 0}\frac{e^{\sin^2 x}-1}{x\tan x}=\lim_{x\to 0}\frac{e^{\sin^2 x}-1}{\sin^2 x}\cdot\frac{\sin^2 x}{x\tan x}=$$
$$=\lim_{x\to 0}\underbrace{\frac{e^{\sin^2 x}-1}{\sin^2 x}}_{\to 1}\cdot\underbrace{\frac{\sin x}{x}}_{\to 1}\cdot\underbrace{\cos x}_{\to 1}=1.$$

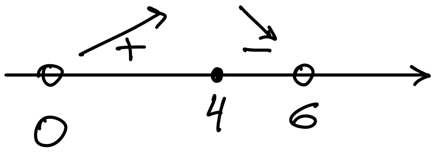
Svar: a) $\frac{9}{2}$ b) $\ln\frac{5}{2}$ c) 1.

4



$x+h=6$; $V=x^2\cdot h \Rightarrow V(x)=x^2(6-x)=$
$=6x^2-x^3$, $0<x<6$.

$V'(x)=12x-3x^2=3x(4-x)=0 \Leftrightarrow$ $x=0$ - ej aktuellt ty $x>0$. eller $x=4$



$V(x)$ är växande för $0<x<4$ och avtagande för $4<x<6$.

Därför $V_{max}=V(4)=32\ (dm^3)$.

Svar: $V_{max}=32\ dm^3$.

5a

$\sin 3x=\sin\left(\frac{\pi}{3}-7x\right) \Leftrightarrow \sin 3x-\sin\left(\frac{\pi}{3}-7x\right)=0 \Leftrightarrow$

$2\sin\frac{3x-\frac{\pi}{3}+7x}{2}\cos\frac{3x+\frac{\pi}{3}-7x}{2}=0 \Leftrightarrow 2\sin\left(5x-\frac{\pi}{6}\right)\cos\left(\frac{\pi}{6}-2x\right)=0$

$\Leftrightarrow 5x-\frac{\pi}{6}=\pi n,\ n\in\mathbb{Z}$ eller $\frac{\pi}{6}-2x=\frac{\pi}{2}+\pi k,\ k\in\mathbb{Z} \Leftrightarrow$

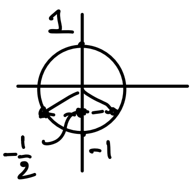
$x=\frac{\pi}{30}+\frac{\pi n}{5},\ n\in\mathbb{Z}$ eller $x=-\frac{\pi}{6}+\frac{\pi k}{2},\ k\in\mathbb{Z}$ - Svar

5b

$2\cos^2 x=2+\sin x \Leftrightarrow 2-2\sin^2 x=2+\sin x \Leftrightarrow$

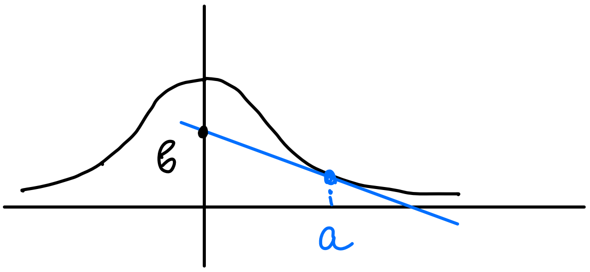
$2\sin^2 x+\sin x=0 \Leftrightarrow 2\sin x\left(\sin x+\frac{1}{2}\right)=0 \Leftrightarrow \sin x=0$

eller $\sin x=-\frac{1}{2} \Leftrightarrow$ $x=\pi n,\ n\in\mathbb{Z}$ eller $x=-\frac{\pi}{6}+2\pi k,\ k\in\mathbb{Z}$



eller $x=\frac{7}{6}\pi+2\pi m,\ m\in\mathbb{Z}$ - Svar.

6. $y = e^{-2x^2}$; $y' = -4x\,e^{-2x^2}$; $y = e^{-2a^2} - 4a\,e^{-2a^2}(x-a)$ – tangenten som tangerar kurvan i punkten $(a,\ e^{-2a^2})$.



Om $x = 0$ får vi $b = e^{-2a^2} + 4a^2 e^{-2a^2}$

Låt $b = f(a) = e^{-2a^2}(1+4a^2)$

$$f'(a) = -4a\,e^{-2a^2}(1+4a^2) + e^{-2a^2}\cdot 8a = -4a\,e^{-2a^2}(1+4a^2-2) =$$
$$= -4a\,e^{-2a^2}(4a^2-1) = 0 \iff a=0,\ a = \pm\tfrac{1}{2}$$

	$-\frac{1}{2}$		0		$\frac{1}{2}$	
$+\ \nearrow$		$-\ \searrow$		$+\ \nearrow$		$-\ \searrow$
	l. max		l. min		l. max	

$\lim_{a\to-\infty} f(a) = \lim_{a\to\infty} f(a) = 0$; $f(0) = 1$

$f(\tfrac{1}{2}) = f(-\tfrac{1}{2}) = \frac{2}{\sqrt{e}} > 1$

Svar: tangenten skär y-axeln i punkten $b \in \left]0, \frac{2}{\sqrt{e}}\right]$.

7 $f(x) = \dfrac{\sin x}{x}$; $f'(x) = \dfrac{x\cos x - \sin x}{x^2} = \dfrac{\cos x\,(x - \tan x)}{x^2} < 0$

då $x \in \left]0, \pi\right[$ ty $\cos x > 0$, $x < \tan x$ för $x \in \left]0,\pi\right[$.

Det betyder att f är strängt avtagande för $x \in \left]0,\pi\right[$

$\Rightarrow$ f har en invers funktion $g = f^{-1}$.

Notera också att f är deriverbar och $f'(x) \neq 0$ för alla $x \in \left]0,\pi\right[$. Enligt satsen är g deriverbar och

$$g'(b) = \frac{1}{f'(a)} \quad \text{där} \quad b = f(a)$$

Om $b = \frac{2}{\pi}$, så gäller $\frac{2}{\pi} = \frac{\sin a}{a} \iff a = \frac{\pi}{2}$.

Därför
$$g'\left(\tfrac{2}{\pi}\right) = \frac{1}{f'\left(\frac{\pi}{2}\right)} = \frac{1}{\dfrac{\frac{\pi}{2}\cos\frac{\pi}{2} - \sin\frac{\pi}{2}}{\left(\frac{\pi}{2}\right)^2}} = -\frac{\pi^2}{4}$$

Svar: $g'\left(\frac{2}{\pi}\right) = -\frac{\pi^2}{4}$.